SIMILARITY PROBLEMS OF THE THEORY OF UNSTEADY CONCENTRATION BOUNDARY LAYER

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Using a new approximate method, transient course of the local and mean diffusion fluxes following a step concentration change on the wall has been obtained for a broad class of steady flow problems.

Investigation of the transient course of diffusion fluxes into the flow past a solid body following a step concentration change on the wall of this body has some interesting applications in the electrochemical determination of diffusivities[1] and in the modelling of heat and mass transfer processes[2]. Within the framework of the simplifications typical for the theory of the concentration boundary layer[2], the theory of transient processes is represented by a linear boundary value problem of the parabolic type. Nevertheless, exact solutions of this problem has been known for only two special transport configurations: for the transport active surface in the front critical region of the flow[1] (the so called configuration with the uniformly accessible surface) and for the transport active semiinfinite plane which is part of the infinite plane passed in parallel by the flow[3] (the so called Leveque configuration). Only incomplete solutions have been known for a number of other problems when the velocity field in the vicinity of the body is given as a solution of the rheodynamic boundary layer[4,5]. These incomplete solutions were obtained by a semiempirical matching of the asymptotic expansions in the real demain or in the complex plane of the Laplace domain[6–9]. These methods *a priori* presume continuous differentiability of the field of concentration in the whole studied time and space domain, a condition which need not be always fulfilled[10,11].

In the present paper it is shown that a suitable similarity transform unites the majority of so far investigated and seemingly diverse problems to a common problem with a single parameter. This one-parameter family of problems is solved approximately by a method providing closed form analytical representation of the transient course of local diffusion fluxes in the vicinity of the wall. A comparison with existing exact solutions indicates that the obtained analytical approximation is sufficiently accurate and closely characterizes even the expected singularities of the exact solution of the equations of unsteady concentration boundary layer.

The Concentration Boundary Layer

Simplified theories of convective diffusion in media with constant transport parameters, which are usually globally referred to as "the theory of the concentration boundary layer", rest on the following assumptions: *a*) Introduction of the local Cartesian coordinates (x, y) (Fig. 1) and corresponding Cartesian velocity components v_x, v_y is associated with neglecting the longitudinal curvature of the surface passed by the flow. This simplification is justified only provided the thickness of the boundary layer is sufficiently small compared to the curvature of the surface. *b*) The velocity field of the components $v_x(x, y)$, $v_y(x, y)$ is represented approximately by a linear profile of longitudinal velocity $v_x = \gamma(x) \,.\, y$, where $\gamma(x)$ is the gradient of longitudinal velocity in the given point, x, of the surface. From here the assumption of liquid adherring to the wall (no slip) and the equation of continuity for planar $(i = 0)$ or axially symmetric $(i = 1)$ flows leads to corresponding quadratic profile of normal velocities $v_y = -r^{-i}(r^i\gamma)'\, z^2/2$, where $r(x)$ designates the distance of the point on the surface from the symmetry axis and where the prime designates the derivative with respect to the coordinate x. The justification of this simplification rests again on the assumption of sufficiently thin concentration boundary layer. *c*) Longitudinal diffusion, represented in the full equations of the convective diffusion by the term $D\partial^2_{xx}c$, is neglected. A consequence of this simplification, which is actually never justified in the whole studied region, is the significant simplification of the problem to a parabolic boundary value problem. On implementing all these assumptions the problem takes the final form

$$D\, \partial^2_{zz}c - \gamma(x)\, y\, (\partial_x c - \tfrac{1}{2}(\ln r^i\gamma)'\, y\, \partial y c) - \partial_t c = 0 \qquad (1)$$

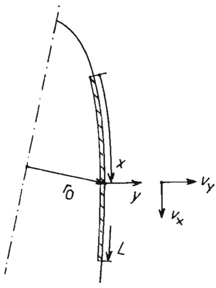

FIG. 1

Local Cartesian coordinates in the theory of the boundary layer with the boundary conditions for $c(t, z, x)$, $t \geqq 0$, $y \geqq 0$, $x > 0$:

$$c = c_0 \quad \text{for} \quad t = 0 \quad \text{or} \quad y \to \infty \tag{2a}$$

$$c = 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad y = 0\,, \tag{2b}$$

A detailed analysis of the conditions of uniqueness of the problem[13,11] points at the necessity of yet another condition ensuring uniform convergence of the solution to the steady state asymptote for $t \to \infty$. One of the possible alternatives of such a condition is

$$t^{3/2}\, \partial^2_{yt} c \to 0 \quad \text{for} \quad t \to \infty\,. \tag{3}$$

The most significant directly measurable characteristics of the transport process are time dependences of the local, J, and mean, $\bar{J}$, diffusion fluxes towards the transport active section, $0 < x < L$, of the surface. For a known concentration field $c(t, y, x)$ these fluxes may be expressed as follows:

$$J(x, t) = D\, \partial_y c|_{y=0} \tag{4}$$

$$\bar{J}(L, t) = \int_0^L J(x, t)\, r^i(x)\, dx \Big/ \int_0^L r^i(x)\, dx\,. \tag{5}$$

Upon introducing the normalized concentration field

$$C(T, Z, \xi) = c(t, y, x)/c_0\,, \tag{6}$$

where

$$Z(y, x) = D^{-1/3} \varkappa(x)\, y\,, \quad T(t, x) = D^{1/3} \varkappa^2(x)\, t\,, \tag{7a,b}$$

$$\varkappa(x) = (r^i(x)\, \gamma(x)/2)^{1/2}\, \xi^{-1/3}(x)\,, \tag{7c}$$

$$\xi(x) = \int_0^x (r^{3i}(s)\, \gamma(s)/8)^{1/2}\, ds \tag{7d}$$

it is possible, with the assumption of "local similarity"

$$\partial_\xi C = 0 \tag{8}$$

to transform the three-dimensional problem in Eqs (1), (2a,b), (3) into a two-dimensional one:

$$0 = \mathbf{L}[C] \equiv \partial^2_{ZZ} C + \tfrac{1}{3} Z^2\, \partial_Z C - (1 - \tfrac{2}{3} \lambda Z T)\, \partial_T C \tag{9}$$

$$C = 1\,, \quad (T = 0 \quad \text{or} \quad Z \to \infty) \tag{10a}$$

$$C = 0\,, \quad (T > 0 \quad \text{and} \quad Z = 0) \tag{10b}$$

$$T^{3/2}\, \partial^2_{ZT} C \to 0\,, \quad (T \to \infty)\,. \tag{10c}$$

The only shape parameter of the considered class of problems is $\lambda = \lambda(\xi)$

$$\lambda = -3\,\frac{\mathrm{d}\ln \varkappa(x)}{\mathrm{d}\ln \xi(x)} = -\frac{3}{2}\,\frac{\partial \ln T(x, t)}{\partial \ln \xi(x)}\,. \tag{11}$$

A detailed analysis shows that the assumption (*8*) is identically fulfilled only if $\partial_\xi \lambda =$ $= 0$, *i.e.* $\lambda =$ const. This condition can be met only for the power-law type functions $\gamma(x)$. For axially symmetric problems the conditions are even more stringent, calling, in addition, for a power-law type course of the function $r(x)$. Generally expressed:

$$\gamma(x) = k_1 x^q\,, \quad r^i(x) = k_2 x^{ip}\,. \tag{12a,b}$$

Substitution of Eqs (*12a,b*) into Eqs (*7a,b*) and (*11*) leads to an explicit expression for the shape parameter

$$\lambda = 2(1 - q)/(2 + q + 3ip)\,. \tag{13}$$

Table I gives a survey of the planar and axially symmetric flows leading to the general similarity formulation of the transient problem. Apart from the familiar situations with corresponding values $\lambda = 0, 1, 2$ the table gives also the situations corresponding to the flow of a non-Newtonian liquid of the flow index $n \in (0; 1\rangle$ past solid bodies under the rheodynamic regime of the laminar boundary layer[4,5,8].

Local and Mean Diffusion Fluxes

In the general case, *i.e.* for the convective configurations not admitting global similarity representation, only the following asymtotic solutions for the initial and the steady state period of the transient process have been known.

Steady state asymptote[2] is represented by the boundary value problem for $C =$ $= C_\infty(Z)$:

$$\partial^2_{ZZ} C + \tfrac{1}{3} Z^2\, \partial_Z C = 0\,, \tag{14}$$

$$C = 1 \quad \text{for} \quad Z = 0\,, \tag{15a}$$

$$C \to 0 \quad \text{for} \quad Z \to \infty\,, \tag{15b}$$

Its solution

$$C_{\infty}(Z) = \beta \int_0^Z \exp\left(-\xi^3/9\right) \mathrm{d}\xi \,, \tag{16}$$

$$\beta = 1 \Big/ \int_0^\infty \exp\left(-\xi^3/9\right) \mathrm{d}\xi = 3^{1/3}/\Gamma(1/3) \doteq 0.538366 \,,$$

leads to the following explicit expression of the local steady state fluxes:

$$J(x, t)\big|_{t=\infty} \equiv J_{\infty}(x) = D^{2/3} c_0 \beta \varkappa(x) \,. \tag{17}$$

Corresponding mean fluxes, according to Eq. (5), may be expressed as

$$\bar{J}(L, t)\big|_{t=\infty} \equiv \bar{J}_{\infty}(L) = D^{2/3} c_0 \beta \bar{\varkappa}(L) \,, \tag{18}$$

where

$$\bar{\varkappa}(L) = \int_0^{\xi(L)} \mathrm{d}\xi^{2/3} \Big/ \int_0^{\xi(L)} \varkappa^{-1} \, \mathrm{d}\xi^{2/3} \,. \tag{19}$$

TABLE I

Review of the similarity flow configurations

Configuration	ip	q	λ
Uniformly accesible configuration (flow in the front critical region)	1 0	1	0
Leveque configuration (simple shear flow)	0	0	1
Blasius configuration (Newtonian flow past a slab)	0	$-1/2$	2
Levich configuration (Newtonian flow past a rotating disc)	1	1	0
Flow of power-law liquid past an immobile wedge [a], see ref.[4], $m \leqq 1$	0	$\dfrac{3m-1}{1+n}$	$\dfrac{4-6m+2n}{1+2n+3m}$
Flow of power-law liquid past a Geiss rotating body [a], see ref.[5], $0 < m < \infty$	m	$\dfrac{3m-1}{1+n}$	$\dfrac{4-6m+2n}{1+2n+(3n+6)m}$
Flow of power-law liquid past an immobile cone [a], see ref.[8], $m \leqq 1$	1	$\dfrac{3m-1}{1+n}$	$\dfrac{4-6m+2n}{4+5n+3m}$

[a] The kinematics is given by the solution of equations of the laminar boundary layer[4,5,8] for a power-law liquid of the flow index $0 < n \leqq 1$; m is a geometrical parameter.

The initial asymptote[2] is represented by the boundary value problem for $C = C_0(T, Z)$

$$\partial^2_{ZZ}C - \partial_T C = 0 \tag{20}$$

$$C \to 1 \quad \text{for} \quad T \to 0 \quad \text{or} \quad Z \to \infty \tag{21a}$$

$$C \to 0 \quad \text{for} \quad T > 0 \quad \text{and} \quad Z \to 0\,. \tag{21b}$$

Its solution

$$C_0(T, Z) = \pi^{-1/2} \int_0^{ZT^{-1/2}} \exp\left(-\xi^2/4\right) \mathrm{d}\xi \tag{22}$$

leads to the following explicit expression of the local instantaneous fluxes:

$$J(x, t)\big|_{t\to 0} \equiv J_0(t) = D^{2/3} c_0 \varkappa(x) (\pi T)^{-1/2} = D^{1/2} c_0 (\pi t)^{-1/2}\,. \tag{23}$$

As the local instantaneous fluxes for $t \to 0$ are independent of the geometrical coordinate x, we have also

$$\bar{J}(L, t)\big|_{t\to 0} = J_0(t)\,. \tag{24}$$

For the globally similar transient problems ($\lambda = \text{const}$) one can express, with a suitable normalization, both the local and instantaneous flows as functions of a single argument, T. For local flows according to Eq. (6), (16) (17) we have

$$J(x, t)/J_\infty(x) = N^*(T) \tag{25}$$

$$N^*(T) = \beta^{-1}\, \partial_Z C(T, Z)\big|_{Z=0}\,. \tag{26}$$

For the mean fluxes one can utilize the proportionalities $\varkappa \sim \xi^{-\lambda/3}$, $T \sim \xi^{-2\lambda/3}$ (see Eq. (11)) and arrive at the following expression

$$N_L(T_L) = \bar{J}(L, t)/\bar{J}_\infty(L) = T_L^{1/\lambda} \int_0^{T_L{}^{-1/\lambda}} N^*(T)\, \mathrm{d}(T^{-1/\lambda})\,, \tag{27}$$

where

$$T_L = T(t, L) = D^{1/3} \varkappa^2(L)\, t\,. \tag{28}$$

It is easily verified that for $T \to \infty$, or $T_L \to \infty$ we have $N^* \to 1$ or $N_L \to 1$. For a comparison of the form of the normalized transient characteristics for different values of the parameters λ it is convenient to normalize the time variable in such a manner as to obtained a common asymptotic course of the transient characteristics also for $T \to 0$ and $T_L \to 0$. For the description of the course of the local or the mean fluxes we therefore chose the newly normalized time variables:

$$\Theta = \pi\beta^2 T, \quad \bar{\Theta} = (1 + \lambda/2)^2 \pi\beta^2 T_L . \tag{29a, b}$$

With this choice the initial and steady state asymptotes of the transient characteristics

$$N(\Theta) = N^*(T), \quad \bar{N}(\bar{\Theta}) = N_L(T_L) \tag{30a,b}$$

intersect at the points $(N, \Theta) = (1, 1)$ or $(\bar{N}, \bar{\Theta}) = (1, 1)$, for now we have, apart from $N(\Theta \to \infty) \to 1$, $\bar{N}(\bar{\Theta} \to \infty) \to 1$ also $N(\Theta \to 0) \to \Theta^{-1/2}$ and $\bar{N}(\bar{\Theta} \to 0) \to \to \bar{\Theta}^{-1/2}$. In the following we shall be concerned with the solution of the given alternative of the problem in terms of the function $N(\Theta)$, $\bar{N}(\bar{\Theta})$. With the known course $N(\Theta)$ one can determine $\bar{N}(\bar{\Theta})$ with the aid of

$$\bar{N}(\bar{\Theta}) = \begin{cases} \dfrac{1}{\lambda}\, \Theta^{1/\lambda} \displaystyle\int_{\Theta}^{\infty} N(s)\, s^{-1-1/\lambda}\, ds\,, & \lambda > 0 \\ N(\Theta)\,, & \lambda = 0 \\ \left(-\dfrac{1}{\lambda}\right) \Theta^{1/\lambda} \displaystyle\int_{0}^{\Theta} N(s)\, s^{-1-1/\lambda}\, ds\,, & \lambda < 0 \end{cases} \tag{31}$$

$$\bar{\Theta} = (1 + \lambda/2)^2\, \Theta \tag{32}$$

following from Eqs (27), (28) and (29a,b).

Approximate Similarity Solution

Current methods of approximate solution of the transient problems of the theory of the transport boundary layer[12,13] may be modified so that the resulting approximation $J(x, t)$ have the asymptotes $t \to 0$ and $t \to \infty$ identical with the exact solution. We shall show that for this purpose it suffices to superimpose the concentration field in the approximate similarity form

$$C(T, Z) = F(\xi)/F(\infty)\,, \quad \xi = a(T)\, Z\,, \tag{33a,b}$$

where the function

$$F(\xi) = 3^{-1/3} \int_0^{\xi^3/9} \exp(-p)\, p^{-2/3}\, dp \tag{34}$$

is an integral of the differential equation $F'' + \frac{1}{3}\xi^2 F' = 0$ of the corresponding steady state boundary value problem, $\xi = Z$. Corresponding concentration gradient on the wall, according to Eq. (33a,b), is clearly given by the relation $\partial_Z C|_{Z=0} = \beta a(T)$ and in the approximation considered we thus have

$$N^*(T) = \beta^{-1} \partial_Z C(T, Z)|_{Z=0} = a(T) \qquad (35)$$

as $F'(0) = 1$ and $1/F(\infty) = \beta$. The requirement of the exact course of the asymptotes according to the approximate solution may thus be expressed by the relationship

$$a(T) \to 1 \quad \text{for} \quad T \to \infty \qquad (36a)$$

$$a(T) \to (\beta^2 \pi T)^{-1/2} \quad \text{for} \quad T \to 0 . \qquad (36b)$$

The overall course of $a(T)$ shall be determined by the method of weighted residuals requiring that for each $T > 0$ the following equality be satisfied,

$$0 = \int_0^\infty \mathbf{L}[C]\, \phi(Z, T)\, \mathrm{d}Z , \qquad (37)$$

where $\mathbf{L}[C]$ is given by Eq. (9) and $\phi(Z, T)$ is an arbitrary weighting function. With the choice $\phi(Z, T) = \psi(a(T)\, Z)$ Eq. (37) reduces to an ordinary differential equation

$$2 \frac{1}{b} \frac{\mathrm{d}a}{\mathrm{d}T} = - \frac{a(a^3 - 1)}{a - \lambda b T} \qquad (38)$$

with an arbitrary constant:

$$b = \frac{2}{3} \int_0^\infty \xi^2 F'(\xi)\, \psi(\xi)\, \mathrm{d}\xi \bigg/ \int_0^\infty \xi F'(\xi)\, \psi(\xi)\, \mathrm{d}\xi . \qquad (39)$$

There exist infinite many weighting functions $\psi(\xi)$ which ensure the value

$$b = \beta^2 \pi \qquad (40)$$

and thus also the asymptotic course of $a(T)$ according to Eq. (36b). One of these function $\psi(\xi)$ is, for instance, very close to $F'(\xi)$.

With the choice $b = \beta^2 \pi$ we have clearly $bT = \beta^2 \pi T = \Theta$ and the similarity approximation of the sought function $N(\Theta) = a(T)$ is, according to Eqs (38), (40) the integral of the following differential equation

$$2\, \mathrm{d}N/\mathrm{d}\Theta = -N(N^3 - 1)/(N - \lambda \Theta) . \qquad (41)$$

The regular integral of the differential equation (41) with the asymptotic condition $N \to \infty$ for $\Theta \to 0$ may be expressed by the quadrature[13]:

$$\Theta = 2(1 - N^{-3})^{2\lambda/3} \int_0^{N^{-1}} (1 - s^3)^{-1-2\lambda/3}\, s\, ds\,. \qquad (42a)$$

The second of the asymptotic conditions, $N \to 1$ for $\Theta \to \infty$, satisfies the singular solution of the differential equation (41):

$$N = 1\,. \qquad (42b)$$

In the examples given in Table I as physically relevant appears the region of the parameters λ delimited by $-2 < \lambda \leqq 4$. For an arbitrary λ from the region thers follows from Eq. (42a) a common asymptotic representation for $\Theta \ll 1$:

$$N(\Theta) \approx \Theta^{-1/2} + (1/5)(1 - \lambda)\,\Theta - (3/200)(1 - \lambda)(1 - 6\lambda)\,\Theta^{5/2} + \\ + (3/4\,400)(1 - \lambda)(1 - 31\lambda)(4 - 3\lambda)\,\Theta^4 \qquad (43)$$

applicable with an accuracy better than 0·2% for $N > 1{\cdot}25$.

From the original differential equation (41) it is apparent that for $\lambda > 0$ one can expect singularity of the solution in the point $N = \lambda\Theta$. From the properties of the integral (42a) there follows that this singularity exists in the point (N_0, Θ_0), where

$$\Theta_0 = N_0/\lambda \qquad (44)$$

clearly $N_0 = 1$ for $0 < \lambda < 1$. For $\lambda > 1$, $N_0 > 1$ is a root of

$$N_0/\lambda - \Theta(N_0) = 0\,, \qquad (45)$$

where $\Theta(N)$ designated the regular branch of the solution according to Eq. (42a). For $\Theta < \Theta_0$, $N(\Theta)$ is given by the regular branch of Eq. (42a) and for $\Theta > \Theta_0$ we have, according to Eq. (42b), $N = 1$. For $0 < \lambda < 1$ thus the function $N(\Theta)$ varies at the point $\Theta = \Theta_0$ continuously, while for $\lambda > 1$, there is a jump at this point from a value $N_0 > 1$ to $N = 1$. Now it is apparent the existence of there regions of values of the parameter λ, $\lambda < 0$, $0 < \lambda < 1$ and $\lambda > 1$, where the solution $N(\Theta)$ possesses qualitatively different properties. To these solutions correspond also different types of representation convenient for practical calculations.

Case I, $\lambda < 0$. The function $N(\Theta)$ is given on the interval $\Theta \in (0; \infty)$ by the regular branch of Eq. (42a). For $\Theta \to \infty$ there is applicable the power-law type asymptote

$$N(\Theta) \approx (1 - ((\Theta - \lambda^{-1})/k)^{3/(2\lambda)})^{-1/3}, \quad (46)$$

where

$$k = 2\int_0^1 (1 - s^3)^{-1-2\lambda/3}\, s\, ds = \Gamma(-2\lambda/3)\,\Gamma(5/3)/\Gamma((1-\lambda)\,2/3). \quad (47)$$

In the limiting case $\lambda = 0$, $N(\Theta)$ may be expressed in the closed form[12,13] as

$$\Theta(N) = \tfrac{1}{3}\ln\left[(N^3 - 1)(N - 1)^{-3}\right] - 2\cdot 3^{-1/2}\left\{\operatorname{arctg}\left[(1 + 2/N)\,.\,3^{-1/2}\right] - \pi/6\right\} \quad (48)$$

or, with sufficient accuracy better than $\pm 0{\cdot}1\%$, by the pair of the asymptotic representations

$$N(\Theta) \approx \begin{cases} \Theta^{-1/2} + 0{\cdot}2\Theta - 0{\cdot}015\Theta^{5/2}, & \Theta \leqq 1 \\ 1 + \varepsilon/(1 - \varepsilon), & 0 \geqq 1 \end{cases} \quad (49a,b)$$

where $\varepsilon = 3^{1/2}\exp(-\pi/(2.3^{1/2}))\exp(-1{\cdot}5\Theta)$.
The functions $N(\Theta)$ and $\bar{N}(\bar{\Theta})$ are identical in the case $\lambda = 0$.

Case II, $0 < \lambda < 1$. The function $N(\Theta)$ is continuous and continuously differentiable, while $N(\Theta) = 1$ for $\Theta \geqq 1/\lambda$. For $\Theta \to (1/\lambda)_-$ one can find an asymptotic expression similar to that in Eq. (*46*)

$$N(\Theta) \approx (1 - ((\lambda^{-1} - \Theta)/k)^{3/(2\lambda)-1/3}. \quad (50)$$

For the limiting case $\lambda = 1$, $N(\Theta)$ is given by the relations

$$N(\Theta) = \begin{cases} \Theta^{-1/2}; & \Theta < 1 \\ 1; & \Theta > 1 \end{cases} \quad (51a,b)$$

with, according to Eq. (*31*), corresponding continuously differentiable course of the mean fluxes

$$\bar{N}(\bar{\Theta}) = \begin{cases} \bar{\Theta}^{-1/2} + \frac{4}{27}\bar{\Theta}; & \bar{\Theta} < \frac{9}{4} \\ 1; & \bar{\Theta} > \frac{9}{4}. \end{cases} \quad (52a,b)$$

Case III, $\lambda > 1$. For $\lambda > 1$ the regular branch $\Theta(N)$ has, according to Eq. (*42a*), a maximum at the point (N_0, Θ_0) while $N_0 = \lambda\Theta_0 > 1$. In the special case of $\lambda = 2$ we have $\Theta_0 = 0{\cdot}54420$. In these cases the solution of the problem, $N(\Theta)$, consists of the already mentioned regular branch for $\Theta < \Theta_0$ and a singular branch $N = 1$

for $\Theta < \Theta_0$. A transition from one branch to another occurs as a jump with an asymptotic course

$$N(\Theta) \approx N_0/(1 - (N_0^3 - 1)^{1/2} \lambda^{-1/2}(1 - \Theta/\Theta_0)^2) \tag{53}$$

for $\Theta \to (\Theta_0)_-$.

Typical courses of the function $N(\Theta)$ and their asymptotic representations are shown in Fig. 2. For practical purposes more relevant are usually the data on the transient development of the mean fluxes, $\bar{N} = \bar{N}(\bar{\Theta})$, where $\bar{\Theta} = (1 + \lambda/2)^2 \Theta$. For $\lambda < 0$ the function $\bar{N}(\bar{\Theta})$ exhibits a singular branch $\bar{N} = 1$ in the region $\bar{\Theta} < (1 + \lambda/2)^2 \Theta_0$ For the regular branches $\bar{N}(\bar{\Theta})$ one can determine by evaluating the following pair of integrals

$$\bar{N} = \begin{cases} N + \Theta^{1/\lambda} \int_N^\infty \Theta^{-1/\lambda}(n)\, dn\,; \quad -2 < \lambda < 0 \\ N - \Theta^{1/\lambda}\left[(N_0 - 1)\, \Theta_0^{-1/\lambda} - \int_N^{N_0} \Theta^{-1/\lambda}(n)\, dn\right]; \quad \lambda < 0 \end{cases} \tag{54a,b}$$

following from the definitions (*31*) and (*32*).

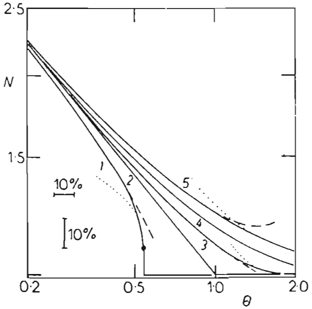

FIG. 2

Local flux. Asymptotix approximations according to Eqs (*43*), (*46*), (*50*) and (*53*) are shown by dotted lines (for $N \to N_0$) or broken lines (for $N \to \infty$). *1* $\lambda = -0.5$, *2* $\lambda = 0$, *3* $\lambda = 0.5$, *4* $\lambda = 1$, *5* $\lambda = 2$

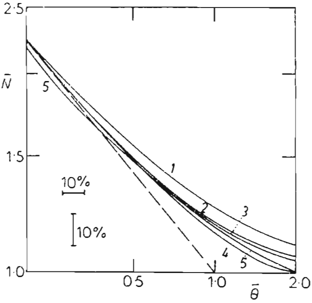

FIG. 3

Mean flux. Same caption as in Fig. 2

For typical values of the parameter λ the functions $\bar{N}(\bar{\Theta})$ are shown in Fig. 3. It is apparent that the courses $\bar{N}(\bar{\Theta})$ do not depend markedly on the parameter λ. Maximum deviations do not exceed 1–3% and are restricted to the region $1 < \bar{\Theta} < 2$.

DISCUSSION

The obtained approximations of the local and mean transient characteristics may deviate from the exact solution of the complete equations of convective diffusion due to the neglected longitudinal diffusion, linear approximation of the velocity profiles or due to the deviations of the applied technique of approximation of the solution of the equations of the concentration boundary layer. The effect of longitudinal diffusion is the strongest in the steady state when the concentration boundary layer has maximum thickness. From the known solutions of the steady state mass or heat transfer at high values of the Prandtl or the Peclet number, however, it is apparent that these errors do not exceed 1% of the value of the Sherwood number[14,15]. For the same reason (small thickness of the concentration boundary layer at high values of the Pr and Pe) the effect of the nonlinearity of the velocity profiles is also negligible. The errors due to the applied technique of approximation, utilizing the similarity technique, can be tested by comparing the known exact solutions of the concentration boundary layer for the configuration of the front critical region[1], $\lambda = 0$ and for the Leveque configuration[3,11], $\lambda = 1$.

The exact and approximate courses of the functions $\bar{N}(\bar{\Theta})$, $N(\Theta)$ are depicted in Fig. 4. In the limiting case $\lambda = 0$ the maximum deviation occurs in the close neighbourhood of the point $\Theta = 1$, where it amounts to 1·5%, while elsewhere it is negligible. For the case $\lambda = 1$ two different exact solutions of the same problem[3,11] are

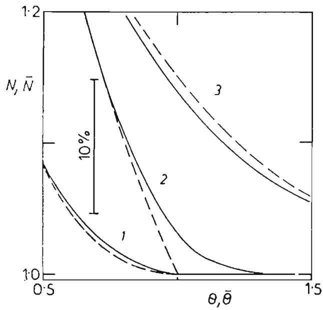

FIG. 4

Comparison of exact and approximate solutions. Solid lines show exact solutions known from literature[1,3], broken lines show their similarity approximations. *1* Mean flux for $\lambda = 1$, *2* local for $\lambda = 1$, *3* mean and local flux for $\lambda = 0$

known. Maximum deviations between these solutions amount to about 2% for $N(\Theta)$ and 0·3% for $\bar{N}(\bar{\Theta})$. The „approximate solutions" according to Eqs (*51a,b*) and (*52a,b*) are identical with one of the exact solutions. It may be therefore expected that the found approximate courses $N(\Theta)$, $\bar{N}(\bar{\Theta})$ are very close to the exact solutions of the complete problem in the whole range of parameters considered $\lambda \in (-1; 2)$ under common conditions of forced convection, *i.e.* at high values of *Pr* and *Pe*. The effect of longitudinal diffusion and the real velocity profiles may be incorporated, while preserving the obtained approximations $N(\Theta)$, $\bar{N}(\bar{\Theta})$, simply by replacing the steady state values $\bar{J}_{\infty}(x)$, according to Eq. (*17*) by the values determined from the exact solution.

A number of attempts to find exact solution of the equations of the concentration boundary layer make use of the methods of the operator calculus[6-9]. While for the case $t \to 0$ one can find without undue difficulty an applicable asymptotic expansion, constructions of the other branch of the solution for $t \to \infty$ encounter always serious difficulties. Superimposed functional series do not ensure convergence and contain free parameters to be determined from the initial conditions, *e.g.* by matching the expansions for $t \to 0$ and $t \to \infty$ in the region of intermediate values of t. Methods of this type presume that the sought solution is firstly analytical in the whole studied region of t and, secondly, approach the steady state asymptote exponentially. According to our approximate solution such a situation occurs in only one case $\lambda = 0$. For $\lambda < 0$ the solution is continuously differentiable, but for $t \to \infty$ approaches the steady state power-law asymptote (see Eq. (*46*)). For $\lambda > 0$ the continuous approximate solution consists of two continuously differentiable branches one of which is identical with the steady state asymptote. Local steadying of the process thus occurs within a finite time. For $\lambda = 1$ the same character exhibits one of the known exact solutions[11]. However, also the second exact solutions[3] exhibits extremely fast transition to the steady state asymptote, for the „decay term" Δ is not exponential, $\Delta \sim \exp(-\alpha T)$, but instead, of the type $\Delta \sim \exp(-\alpha T^3)$. The justification of the

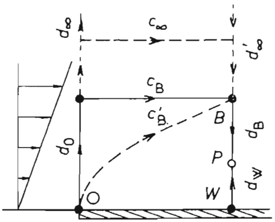

FIG. 5

Scheme of spreading of the concentration signal. *O* Origin, *W* wall, *B* boundary of the steady state concentration boundary layer, *P* point within the boundary layer; d_0, d_∞, d'_∞, d_B, d_W spread of the signal by diffusion; c_∞, c_B spread of the signal by convection; c'_B combined spread of the signal toward the limits of the boundary layer methods of solution in refs[6–9] thus appears suspect excepting the case[7] of $\lambda = 0$. In error is probably the assumption of the convergence of the superimposed functional series[6], already in the trivial case $t = \infty$, in the expression for the steady state value of the diffusion flux.

The fact that the exact solution of the transient problems of the concentration boundary layer may possess also a singular character, with the transition to the steady state regime within a finite time[10,11], seems to have a simple physical explanation. While the diffusion terms in the equation of convective diffusion represent an infinite fast spread of the concentration signal in the appropriate direction, the convective terms represent the spread of the signal by the finite velocity of the flow. The equations of the concentration boundary layer, ignoring the longitudinal diffusion, imply only infinitely fast spreading of the concentration signal only in the direction perpendicular to the passed surface, see Fig. 5. In the longitudinal direction the signal spreads only by local velocity of the flow.

Let us now consider the process of steadying the regime in a given point P as a response to the signal from the surrounding. At the time $t = 0$ part of the surface of the wall, $0 < x < L$, changes in a stepwise manner the concentration, while the front part of the surface $x < 0$ maintains its original concentration conditions. An „Observer" at the point P notices these changes instantaneously starting from the point W on the wall, while the signals from the surroundings do not indicate any change. He reacts as if there was only a concentration change on the whole surface and not only the part $x > 0$. The complete model of such a process is the penetration asymptote according to Eqs (*20*)–(*22*). Because the concentration changes toward the steady state regime take place under such conditions very rapidly, the local steady state value of the concentration at the point P may be realized in a finite time corresponding to the minimum time that the convective signal takes to travel the distance from the point 0 to the point P along the trajectory falling totally into or on the limits of the concentration boundary layer. This qualitative concept agrees very well with the computed and experimental time for reaching the steady state. For $\lambda = 1$ one of the existing solutions[3] on the one hand predicts infinite time to reach the steady state, but, on the other hand, with insignificant deviations from the steady state value (Fig. 4). This is given by the fact that the equation of the concentration boundary layer with the linear velocity profile admit infinite velocity, $v_x \to \infty$, at the infinite distance from the surface, $y \to \infty$. A certain residual convective signal thus spreads with infinite velocity but an infinite decay of intensity.

The phenomena for $y \to \infty$ from which the concept of the concentration boundary layer seemingly totally abstracts, thus do have a certain effect on the form of the transient characteristics. In the region $y \to \infty$ the effect of the neglected longitudinal diffusion is, of course, dominant and must therefore, according to the presented qualitative ideas, exercise a major effect on the shape of the transition characteristics in the region of intermediate times and on the character of the concentration field. Nevertheless, from the practical point of view the resulting difference shall be insignificant as suggests the surprisingly good agreement of the two exact solutions of the equations of the concentration boundary layer for the case $\lambda = 1$.

LIST OF SYMBOLS

c	concentration of transport active component
c_0	concentration in bulk liquid
C	normalized concentration field
D	diffusion coefficient
i	index of the planar ($i = 0$) or axial ($i = 1$) symmetry
J	local diffusional flux
J_∞	steady state value of J
$\bar{J}$	diffusional flux averaged over the transport active part surface
$\bar{J}_\infty$	steady state value of $\bar{J}$
k	constant defined in Eq. (*47*)
L	length of transport active part of surface
L	differential operator
N, $\bar{N}$	normalized diffusional fluxes
n	flow index
p, q	parameters of velocity field
$r(x)$	profile of axially symmetric body
t	time
T, T_1	time-space similarity variables
v_x, v_y	components of the velocity field
x, y	local Cartesian coordinates
Z	spatial similarity variable
β	numerical constant (Eq. (*16*))
γ	velocity gradient on the surface of the body
Θ, $\bar{\Theta}$	renormalized time-space similarity variables
$\varkappa$, $\bar{\varkappa}$	parameters of similarity transformation
λ	shape parameter
ξ	modified longitudinal coordinate

REFERENCES

1. Nisancioglu K., Newman J.: J. Electroanal. Chem. Interfacial Electrochem. *50*, 23 (1974).
2. Fahidy T. Z., Mohanta S. in book: *Advances in Transport Processes* (A. S. Mujumdar, Ed.) Vol. I, p. 83. Wiley East, New Delhi 1980.
3. Soliman M., Chambre P. L.: Int. J. Heat Mass Transfer *10*, 169 (1967).
4. Shul'man Z. P.: *Convective Heat- and Mass Transfer in Rheologically Complex Liquids.* Energia, Moscow 1975.
5. Wein O., Mitschka P., Wichterle K.: *Rotational Flows od Non-Newtonian Liquids.* Academia, Prague 1981.
6. Chen J. L. S., Chao B. T.: Int. J. Heat Mass Transfer *13*, 1101 (1970).
7. Riley N.: Acta Mech. *3*, 285 (1969).

8. Shul'man Z. P., Pokryvaylo N. A., Sobolevskii A. S., Jushkina T. V.: Inzh.-Fiz. Zh. *30*, 436 (1976).
9. Shul'man Z. P., Pokryvaylo N. A., Jushkina T. V.: Inzh.-Fiz. Zh. *24*, 992 (1973).
10. Chambre P. L., Soliman M.: Int. J. Heat Mass Transfer *12*, 1301 (1969).
11. Wein O.: This Journal *46*, 3209 (1981).
12. Goodman T. R.: J. Heat Transfer *84*, 347 (1962).
13. Bruckenstein S., Prager S.: Anal. Chem. *39*, 1161 (1967).
14. Smyrl W. H., Newman J.: J. Elektrochem. Soc. *118*, 1079 (1971).
15. Nagasue H.: Int. J. Heat Mass Transfer *24*, 1823 (1981).

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